SMALL PERTURBATIONS OF PLANE UNSTEADY MOTION OF AN IDEAL INCOMPRESSIBLE FLUID WITH A FREE BOUNDARY IN THE SHAPE OF AN ELLIPSE

V. V. Pukhnachev

UDC 532.517

The problem about small perturbations is solved explicitly. An investigation of the behavior of the solution as $t \to \infty$ shows its boundedness in a weak "potential" metric. Meanwhile, the perturbation vector of the free boundary of the ellipse grows without limit with time.

1. FORMULATION OF THE PROBLEM

It is known that the equations of plane potential motion of an ideal incompressible fluid admit the exact solution

$$u = \dot{\tau}\xi = \frac{\dot{\tau}}{\tau}, \qquad v = -\frac{\dot{\tau}}{\tau^2}\eta = -\frac{\dot{\tau}}{\tau}y \tag{1.1}$$
$$p = -{}^1\!/_2\, \tau\ddot{\tau}\ (\zeta^2 + \eta^2 - 1)$$

Here ξ, η are the Lagrange coordinates, and x, y the Euler coordinates, t is the time, the dot at the location of a prime indicates differentiation with respect to t, and the function τ (t) is given by the relationship

$$\int_1^{\tau} \sqrt{\rho^4 + 1}\, \frac{d\rho}{\rho^2} = kt \qquad (k = \text{const} > 0) \tag{1.2}$$

The solution (1.1) admits of simple interpretation: at t = 0 the velocity field

$$u_0 = {}^1\!/_2 \sqrt{2}\, k\xi = {}^1\!/_2 \sqrt{2}\, kx, \qquad v_0 = -{}^1\!/_2 \sqrt{2}k\eta = -{}^1\!/_2 \sqrt{2}k$$

is given in the circle $\Omega\,(\xi^2 + \eta^2 < 1)$.

As t grows the circle Ω is deformed into an ellipse Ω_t, whose semimajor axis is $\tau \to \infty$, and the semiminor axis is $\tau^{-1} \to 0$. The boundary of the ellipse remains free for all $t \geq 0$.

Let us examine another solution of the equations of plane potential motion in the same domain Ω of the Lagrange coordinate plane, but with an altered initial potential

$$\varphi_0^*\,(\xi, \eta) = \varphi_0\,(\xi, \eta) + \Phi_0\,(\xi, \eta), \qquad \Delta\Phi_0 = 0$$
$$\varphi_0 = 2^{-3/2} k\,(\xi^2 - \eta^2)$$

Here φ_0 is the value of the main flow potential (1.1) at t = 0, and Φ_0 is the initial perturbation potential. Assuming the initial perturbation Φ_0 small, we can study the problem of evolution of small perturbations in a linear formulation. The equations of small potential motion perturbations with a free boundary were derived by L. V. Ovsyannikov [1]. In the case of the fundamental solution (1.1) they are

$$\frac{1}{\tau^2}\Phi_{\xi\xi} + \tau^2\Phi_{\eta\eta} = 0 \qquad (\xi^2 + \eta^2 < 1,\ t \geqslant 0) \tag{1.3}$$

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 4, pp. 53–62, July–August, 1971. Original article submitted February 9, 1971.

$$\Phi_t = -\frac{2k^2\tau^4}{(\tau^4+1)^2}\int_0^t \left(\frac{1}{\tau^2}\xi\Phi_\xi + \tau^2\eta\Phi_\eta\right)dt \qquad (1.4)$$

$$(\xi^2+\eta^2=1,\ t>0)$$

The initial condition

$$\Phi(\xi,\eta,0) = \Phi_0(\xi,\eta), \qquad \Delta\Phi_0 = 0 \qquad (1.5)$$

should be added to these equations.

The solution of the problem (1.3)-(1.5) as $t\to\infty$ is investigated below; the results permit making deductions on the stability of the main flow (1.1) relative to the potential perturbations.

Let us note that there is not presently any general approach to the study of the stability of unsteady fluid motions with a free boundary. Each such stability problem must be examined individually. Up to now only plane problems have been investigated in which the free boundaries are lines or circles [2, 1] (see [3], also). By analogy with the motion examples considered, L. V. Ovsyannikov expressed the hypothesis that the motion described by (1.1) will be unstable. The solution of the problem (1.3)-(1.5) is constructed explicitly below. An analysis of the asymptotic of the solution as $t\to\infty$ confirms this hypothesis.

2. CONSTRUCTION OF THE SOLUTION OF THE PROBLEM

Let us go over to the independent variable τ instead of t in (1.3), (1.4). By virtue of (1.2) the correspondence between τ and t is one-to-one for $\tau \ge 1$ $(t\ge 0)$. Let us replace the desired function

$$\Phi(\xi,\eta,t) = \tau\left(\frac{2}{\tau^4+1}\right)^{3/4} w(\xi,\eta,\tau)$$

and let us differentiate the transformed equation (1.4) with respect to τ. Consequently, we obtain in place of (1.3), (1.4)

$$\frac{1}{\tau^2}w_{\xi\xi} + \tau^2 w_{\eta\eta} = 0 \qquad (\xi^2+\eta^2<1,\ \tau\geqslant 1) \qquad (2.1)$$

$$w_{\tau\tau} + \frac{2}{\tau^4+1}\left(\frac{1}{\tau^2}\xi w_\xi + \tau^2\eta w_\eta\right) + r(\tau)w = 0 \qquad (2.2)$$

$$(\xi^2+\eta^2=1,\ \tau>1)$$

Here

$$r(\tau) = -\frac{2\tau^8+7\tau^4+2}{\tau^2(\tau^4+1)^2}$$

Let us write down the initial conditions for these equations. From the definition it follows that

$$\Phi = w, \quad \Phi_\tau = w_\tau - {}^1/_2 w \quad \text{at} \quad \tau = 1$$

According to (1.4), (1.5) we have

$$\Phi = \Phi_0, \quad \Phi_\tau = 0 \quad \text{at} \quad \tau = 1$$

Hence

$$w = 2w_\tau = \Phi_0(\xi,\eta) \quad \text{at} \quad \tau = 1 \qquad (2.3)$$

The function Φ_0 is harmonic in a circle Ω, hence it is sufficient to give Φ_0 on the circle $\Gamma(\xi^2+\eta^2=1)$

$$\Phi_0|_\Gamma = \psi(\theta), \quad 0 - \operatorname{arc\,tg}(\eta/\xi)$$

Let us consider the periodic function ψ given by its Fourier series

$$\psi(\theta) = \sum_{n=1}^{\infty}(f_n\cos n\theta + g_n\sin\theta) + f_0$$

Without limiting the generality, we can assume $f_0 = 0$. Indeed, if $f_n = g_n = 0$ $(n = 1, 2, \ldots)$, then the unique solution of the problem (2.1)-(2.3) is $\Phi_0 = f_0 = \text{const}$.

Now, let us formulate the problem (2.1)-(2.3) in terms of the boundary function z

$$z(\theta,\tau) = w(\xi,\eta,\tau)|_\Gamma$$

It is clear that the value of z permits unique determination of w in the circle Ω as a solution of the Dirchlet problem for (2.1). Let us introduce an operator K acting according to the rule: a function $w(\xi, \eta, \tau)$, is found by means of $z(\theta, \tau)$, which satisfies (2.1) in the circle Ω and the condition $w|_\Gamma = z$, and then

$$K(\tau) z = \tau^{-2}\xi w_\xi + \tau^2 \eta w_\eta \quad \text{at} \quad \xi^2 + \eta^2 = 1 \tag{2.4}$$

The problem (2.1)-(2.3) can be considered as a Cauchy problem for an equation with nonlocal operator K

$$z_{\tau\tau} + \frac{2}{\tau^4+1} K(\tau) z + r(\tau) z = 0 \tag{2.5}$$

$$z(\theta + 2\pi, \tau) = z(\theta, \tau) \tag{2.6}$$

$$z(\theta, 1) = 2z_\tau(\theta, 1) = \psi(\theta) \tag{2.7}$$

Equation (2.5) is (2.2) rewritten in a new notation, and (2.7) are the initial conditions (2.3) written on the boundary Γ of the circle Ω.

As has been remarked above, without limiting the generality we can consider the mean value $\psi(\theta)$ to be zero. Let us show that then for $\tau > 1$

$$\chi(\tau) \equiv \int_0^{2\pi} z(\theta, \tau)\, d\tau = 0 \tag{2.8}$$

On the basis of (2.7) we obtain $\chi(1) = \chi_\tau(1) = 0$. Let us integrate (2.5) with respect to θ between 0 and 2π. Taking into account that

$$\int_0^{2\pi} Kz d\theta = \int_\Gamma (\tau^{-2}\xi w_\xi + \tau^2\eta w_\eta)\, d\Gamma = \int_\Omega (\tau^{-2} w_{\xi\xi} + \tau^2 w_{\eta\eta})\, d\Omega = 0$$

we find the following equation for χ:

$$\chi_{\tau\tau} + r(\tau)\chi = 0$$

Thus the function χ is a solution of the homogeneous Cauchy problem for the linear equation, which indeed yields (2.8).

Let $L_2'(0, 2\pi)$ denote the subspace of the Hilbert space $L_2(0, 2\pi)$ generated by functions with zero mean value. It follows from (2.8) that if the solution $z(\theta, \tau)$ of the problem (2.5)-(2.7) belongs to $L_2(0, 2\pi)$ for fixed $\tau > 1$, then it also belongs to $L_2'(0, 2\pi)$.

Let us examine a number of properties of the operator K. Let us fix the value $\tau \geq 1$. The operator K is not defined in the whole space $L_2'(0, 2\pi)$. It is, however, defined in all trigonometric polynomials with zero free member, i.e., in a compact set in $L_2'(0, 2\pi)$. Thus, the unbounded operator K has a compact domain of definition D(K) in $L_2'(0, 2\pi)$. Let us show that it is symmetric and positive-definite. Indeed, if $\bar{z}, z \in D(K)$, then

$$\int_0^{2\pi} \bar{z} K z d\theta = \int_\Gamma \bar{w}(\tau^{-2}\xi w_\xi + \tau^2\eta w_\eta)\, d\Gamma = \int_\Gamma w(\tau^{-2}\xi \bar{w}_\xi + \tau^2\eta \bar{w}_\eta)\, d\Gamma = \int_0^{2\pi} z K\bar{z} d\theta$$

$$\int_0^{2\pi} z K z d\theta = \int_\Gamma w(\tau^{-2}\xi w_\xi + \tau^2\eta w_\eta)\, d\Gamma = \int_\Omega (\tau^{-2} w_\xi^2 + \tau^2 w_\eta^2)\, d\Omega$$

QED. (The definition (2.4) of the operator K and the Green's formulas for the solutions w, $\bar{w}$ of (2.1) were used in writing these latter equalities.) It follows from the listed properties of the operator K that it admits of self-adjoint expansion [4], which we again denote to be K. Moreover, the operator inverse to K is completely continuous. This permits the conclusion that the operator K has a complete system of eigenfunctions in $L_2'(0, 2\pi)$ (for any $\tau = \text{const} \geq 1$). It turns out that they can be written down explicitly.

Proposition 1. For any natural n the functions $\cos n\theta$ and $\sin n\theta$ are eigenfunctions of the operator K, to which the eigennumbers

$$\lambda_n = n\, \text{th}\, ns(\tau), \quad \mu_n = n\, \text{cth}\, ns(\tau), \quad s(\tau) = \text{ar th}\, \tau^{-2}$$

correspond.

Proof. Let us go over to elliptical coordinates p, q in (2.1), (2.4) by means of the formulas

$$\operatorname{ch} p \cos q = \frac{\tau^2 \xi}{\sqrt{\tau^4 - 1}}, \qquad \operatorname{sh} p \sin q = \frac{\eta}{\sqrt{\tau^4 - 1}} \tag{2.9}$$

By means of (2.9) the circle Ω is mapped into a circle given in the polar coordinates p, q by the relationships

$$p < s = \operatorname{ar\,th} \tau^{-2}, \qquad 0 \leqslant q < 2\pi$$

Equations (2.1) go over into the equations

$$W_{pp} + W_{qq} = 0, \qquad W(p, q, \tau) = w(\xi, \eta, \tau)$$

The relationship (2.4) becomes

$$Kz = \frac{\partial W}{\partial p} \quad \text{for} \quad p = s \qquad (z = W|_{p=s}) \tag{2.10}$$

(it is kept in mind that the right side is here expressed in terms of θ, τ).

The equation $W_{pp} + W_{qq} = 0$ has the particular solutions

$$\begin{aligned} W_{1n} &= (\operatorname{ch} ns)^{-1} \operatorname{ch} np \cos nq \\ W_2 &= (\operatorname{sh} ns)^{-1} \operatorname{sh} np \sin nq \end{aligned} \qquad (n = 1, 2, \ldots) \tag{2.11}$$

Solutions of (2.1) regular in the circle Ω correspond to these solutions. Hence

$$W_{1n} = \cos nq, \quad W_{2n} = \sin nq \quad \text{for} \quad p = s$$
$$\frac{\partial W_{1n}}{\partial p} = n \operatorname{th} ns \cos nq, \qquad \frac{\partial W_{2n}}{\partial p} = n \operatorname{cth} ns \sin nq$$

Assuming $z_{1n} = \cos n\theta$, $z_{2n} = \sin n\theta$, and taking into account that $q = \theta = \arctan \eta/\xi$ for $p = s$ we obtain from these latter equalities and (2.10) that

$$Kz_{1n} = \lambda_n z_{1n}, \quad Kz_{2n} = \mu_n z_{2n}.$$

Proposition 1 is proved.

The fact that the eigenfunctions of the operator K are independent of τ is quite important. This affords the possibility of separating variables in (2.5). Let us seek the solution of this equation as an eigenfunction series

$$z(\theta, \tau) = \sum_{n=1}^{\infty} [f_n a_n(\tau) \cos n\theta + g_n b_n(\tau) \sin n\theta] \tag{2.12}$$

Here f_n, g_n are coefficients of the Fourier series expansion of the initial function $\psi(\theta)$, and $a_n(\tau)$, $b_n(\tau)$ are functions to be determined.

The solution (2.12) will satisfy the condition (2.6) if we assume

$$a_n(1) = b_n(1) = 2a_n'(1) = 2b_n'(1) = 1 \tag{2.13}$$

(the prime denotes differentiation with respect to τ). Substituting (2.12) into (2.5) results in a decomposing system of ordinary differential equations for the functions

$$\begin{aligned} a_n'' + \left[\frac{2n \operatorname{th} ns(\tau)}{\tau^4 + 1} + r(\tau)\right] a_n &= 0 \\ b_n'' + \left[\frac{2n \operatorname{cth} ns(\tau)}{\tau^4 + 1} + r(\tau)\right] b_n &= 0 \end{aligned} \tag{2.14}$$

Therefore, seeking the solution of the nonlocal Cauchy problem (2.5)-(2.7) reduces to solving the Cauchy problem (2.13), (2.14) for ordinary equations. Knowledge of the function z permits us to write down the solution of the problem (2.1)-(2.3) without difficulty, and this means the desired function $\Phi(\xi, \eta, \tau)$ also. Indeed, w is a solution of the Dirichlet problem for (2.1) with the condition $w_\Gamma = z$.

The quantity τ enters into (2.1) as a parameter; hence, to determine w it is sufficient to solve the given problem $z = z_{1n} = \cos n\theta$ and $z = z_{2n} = \sin n\theta$ (n is natural and fixed). But, as is clear from the proof of Proposition 1, the solution of such a problem is given by (2.11). Let us formulate the final result. The solution of the problem (1.3)-(1.5) is

$$\Phi = \tau\left(\frac{2}{\tau^4+1}\right)^{3/4} \sum_{n=1}^{\infty}\left[f_n a_n(\tau)\frac{\operatorname{ch} np(\xi,\eta,\tau)}{\operatorname{ch} ns(\tau)}\cos nq(\xi,\eta,\tau) + g_n b_n(\tau)\frac{\operatorname{sh} np(\xi,\eta,\tau)}{\operatorname{sh} ns(\tau)}\sin nq(\xi,\eta,\tau)\right] \tag{2.15}$$

Here the functions p, q $(0 \le p \le s,\ 0 \le q < 2\pi)$ are found from (2.9), $s = \operatorname{ar\,th} \tau^{-2}$, τ is related to t by means of (1.2). The functions $a_n(\tau)$, $b_n(\tau)$ are a solution of (2.14) with the conditions (2.13).

Let us note that the solution constructed for the problem (1.3)-(1.5) is unique. The uniqueness theorem for more general problems of the small perturbations of unsteady ideal fluid motion with a free boundary has been proved in [1, 3].

3. ASYMPTOTIC BEHAVIOR OF THE SOLUTION

To obtain the asymptotic solution of the problem (1.3)-(1.5) for large t it is necessary to know the behavior of solutions of the Cauchy problems (2.13), (2.14) as $\tau \to \infty$, since $\tau = kt[1 + O(t^{-4})]$ for large t in conformity with (1.2).

Let n be fixed. Let us examine (2.14) for large τ. Taking into account that

$$s = \tau^{-2} + O(\tau^{-6}), \quad r = -2\tau^{-2} + O(\tau^{-6}) \quad \text{for} \quad \tau \to \infty$$

we have

$$a_n'' + [-2\tau^{-2} + O(\tau^{-6})]\,a_n = 0, \quad b_n'' + O(\tau^{-6})\,b_n = 0 \quad \text{for} \quad \tau \to \infty$$

We hence find the asymptotic representation of two linearly independent solutions of each of the equations (2.14) as $\tau \to \infty$

$$\begin{aligned} a_{n1} &= \tau^2[1 + O(\tau^{-4})], & a_{n2} &= \tau^{-1}[1 + O(\tau^{-4})] \\ a_{n1}' &= 2\tau[1 + O(\tau^{-4})], & a_{n2}' &= -\tau^{-2}[1 + O(\tau^{-4})] \\ b_{n1} &= \tau[1 + O(\tau^{-4})], & b_{n2} &= 1 + O(\tau^{-4}) \\ b_{n1}' &= 1 + O(\tau^{-4}), & b_{n2}' &= O(\tau^{-5}) \end{aligned} \tag{3.1}$$

If we limit ourselves to an examination of an individual harmonic (this means that all the coefficients f_n, g_n in (2.12) are zero except for one), then the last formulas are sufficient to prove the boundedness of the solution Φ of the problem (1.3)-(1.5) as $t \to \infty$.

Indeed, in this case it follows from (3.1) that $z(\theta, \tau) = O(t^2)$ uniformly in θ as $\tau \to \infty$. Since

$$\Phi|_\Gamma = \tau\left(\frac{2}{\tau^4+1}\right)^{3/4} z(\theta, \tau) \tag{3.2}$$

then the boundedness of Φ_Γ hence results as $t \to \infty$. By virtue of the maximum principle for (1.3), the function Φ will be bounded as $t \to \infty$ for any $\xi, \eta \in \Omega$.

Now, let the initial function $\psi(\theta)$ be an arbitrary element of $L_2(0, 2\pi)$. It turns out that even in this case the solution of the problem (1.3)-(1.5) is bounded in the following sense:

$$\int_\Gamma \Phi|_\Gamma^2 d\Gamma \le C_0 \|\psi\|_{L_2}^2 \qquad (t \ge 0) \tag{3.3}$$

$$\|\psi\|_{L_2} = \left(\int_0^{2\pi} \psi^2(\theta)\, d\theta\right)^{1/2} = \left[\pi \sum_{n=1}^{\infty} (f_n^2 + g_n^2)\right]^{1/2}$$

Here C_k, k = 0, 1, 2, ... denote positive constants. The proof of the estimate (3.2) is based on the following proposition.

Proposition 2. The solution of each of Eqs. (2.14) with the initial conditions (2.13) satisfies the inequalities

$$|a_n(\tau)| \le C_1 \max(\tau, \tau^2/\sqrt{n}), \quad |b_n(\tau)| \le C_2\tau \tag{3.4}$$

$$|a_n'(\tau)| \le C_1 \max(\sqrt{n}/\tau, \tau/\sqrt{n}), \quad |b_n'(\tau)| \le C_2 \max(\sqrt{n}/\tau, 1) \tag{3.5}$$

$$(\tau \ge 1,\ n = 1, 2, 3, \ldots)$$

The proof of this proposition is elucidated in Sec. 4.

The inequality (3.3) is a simple consequence of (3.2) and the estimates (3.4). The results obtained about the boundedness of $\|\Phi_\Gamma\|_{L_2}$ as $t \to \infty$ can be treated as the stability, in a linear approximation, of the main solution in a potential metric.

It should be noted that in the particular case when all $f_n = 0$, there results from (3.2), (3.4) that $\| \Phi|_{\Gamma} \|_{L_2} = O(t^{-1})$ as $t \to \infty$. Moreover, if $g_n = 0$ for odd n, then the solution Φ defined by (2.15) will be an even function of ξ and an odd function of η.

The solution even in ξ describes the motion with an impermeable wall $\xi = 0$, hence, such motion is asymptotically stable in a potential metric relative to perturbations odd in η.

Together with (3.2), (2.12), the inequalities (3.5) permit the proof that the following estimate for the derivative Φ_t is true:

$$\int_{\Gamma} \Phi_t^2|_{\Gamma}\, d\Gamma \leqslant C_3 \sum_{n=1}^{\infty} n\,(f_n^2 + g_n^2) \tag{3.6}$$

under the condition that the series in the right side converges. The demand for its convergence is equivalent to the initial function $\psi(\theta)$ belonging to the Sobolev-Slobodetskii space $W_2^{1/2}(0, 2\pi)$.

By raising the smoothness of $\psi(\theta)$, estimates can be obtained for higher order derivatives of Φ. In particular, if $\psi \in W_2^1(0, 2\pi)$, where W_2^1 is the Sobolev space [4], then $\Phi_{tt}|_{\Gamma}$, $\Phi_{\xi}|_{\Gamma}$, $\Phi_{\eta}|_{\Gamma}$ belong to $L_2(0, 2\pi)$ for fixed t, where their norms in L_2 are bounded for all $t > 0$. If $\psi \in W_2^2(0, 2\pi)$, the the functions $z_{\tau\tau}$, Kz in (2.5) are continuous in θ, τ, and this equation is satisfied in the classical sense.

From the physical viewpoint it is interesting to obtain estimates of the velocity field originating during perturbation of the main flow (1.1). In conformity with [1], the projections U, V of the velocity vector perturbations are calculated by means of the formulas

$$U = \frac{\partial}{\partial t}\left(\tau \int_0^t \frac{1}{\tau^2}\Phi_{\xi} dt\right), \quad V = \frac{\partial}{\partial t}\left(\frac{1}{\tau}\int_0^t \tau^2 \Phi_{\eta} dt\right) \tag{3.7}$$

For simplicity, let us limit ourselves to the examination of one harmonic. Let the subscript c henceforth denote the solution corresponding to the initial function $\psi = f_n \cos n\theta$, and the subscript s the solution corresponding to $\psi = g_n \sin n\theta$. The analysis of the asymptotic U, V as $t \to \infty$ starts from (3.7), (2.15), (3.1). This analysis requires many computations and is not presented here. It shows that for $x, y \in \Omega_t$ and $t \to \infty$ the quantities U_s, U_c increase linearly in t near the ends of the major axis of the ellipse Ω_t, V_s is bounded, and $V_c \to \infty$.

The results presented above favor the stability of the main flow in a linear approximation if the norm of the potential perturbation or its derivatives in L_2 is taken as the measure of stability. However, if the stability is judged by the deviation of the free boundary from its unperturbed state, then the motion (1.1) should be acknowledged unstable. It is most convenient to characterize the perturbation vector of the boundary of the ellipse Ω_t by its normal component R to Γ_t (see [1, 3] for the definition and geometric meaning). The quantity for the main flow (1.1) is given by the equality

$$R = -\frac{(\tau^4+1)^2}{2k^2\tau^2(\cos^2\theta + \tau^4 \sin^2\theta)}\Phi_t|_{\Gamma} \tag{3.8}$$

Using (3.2) and (2.12), a representation for R can be obtained as a Fourier series with coefficients expressed in terms of $a_n(\tau)$, $b_n(\tau)$. As before, let us limit ourselves to an examination of individual harmonics, and we find:

for the c-solution

$$R_c = -\frac{f_n(\tau^4+1)^{3/2}\cos n\theta}{2k(\cos^2\theta + \tau^4\sin^2\theta)}\frac{d}{d\tau}\left[\tau\left(\frac{2}{\tau^4+1}\right)^{3/4} a_n(\tau)\right] \tag{3.9}$$

for the s-solution

$$R_s = -\frac{g_n(\tau^4+1)^{3/2}\sin n\theta}{2k(\cos^2\theta + \tau^4\sin^2\theta)}\frac{d}{d\tau}\left[\tau\left(\frac{2}{\tau^4+1}\right)^{3/4} b_n(\tau)\right] \tag{3.10}$$

Since n is fixed in these relationships, it is then sufficient to use (3.1) to obtain the asymptotics R_c, R_s as $t \to \infty$; consequently, we obtain as $t \to \infty$

$$\begin{aligned} R_c &= \frac{\tau^2 \cos n\theta}{\cos^2\theta + \tau^4 \sin^2\theta}[f_n\beta_n + O(\tau^{-1})] \quad (\beta_n = \text{const}) \\ R_s &= \frac{\tau^4 \sin n\theta}{\cos^2\theta + \tau^4 \sin^2\theta}[g_n\gamma_n + O(\tau^{-1})] \quad (\gamma_n = \text{const}) \end{aligned} \tag{3.11}$$

Let $\varepsilon > 0$ be fixed. It is seen from (3.11) that outside the zone $|\theta| < \varepsilon$, $|\pi - \theta| < \varepsilon$ the estimates $R_c = O(t^{-2})$, $R_s = O(1)$ are valid as $t \to \infty$ (let us recall that $\tau \sim kt$ for large t). Therefore, the instability of the free boundary is manifest near the points $\theta = 0$, $\theta = \pi$ of the circle Γ in the Lagrange coordinate plane.

Let us examine the behavior of R_c, R_s as $t \to \infty$, $\theta \to 0$. (The analysis of the case $\theta \to \pi$ is analogous). According to (3.11), if $t|\theta| = \text{const}$ as $t \to \infty$, then the quantity R_c still remains bounded. If $t^{1+\sigma}|\theta| = \text{const}$, $0 \le \delta \le 1$, as $t \to \infty$, then $R_c \sim t^{2\delta}$.

The maximum growth $R_c \sim t^2$ is observed in the domain $t^2|\theta| < \text{const}$. Hence, R_c retains its sign near $\theta = 0$.

The instability of the free boundary for the s-solution is developed differently. In this case we have $R_s \sim t^{\delta}$ under the condition $t^{\delta}|\theta| = \text{const}$ $(0 \le \delta \le 2)$, when $t \to \infty$. The greatest growth $R_s \sim t^2$ occurs when θ and t are related by the relationship $t^2|\theta| \sim 1$ as $t \to \infty$, where R_s changes sign in the neighborhood of $\theta = 0$; R_s is bounded if $|\theta| = O(t^{-4})$, $R_s = 0$ for $\theta = 0$.

It is interesting to estimate the size of the free boundary instability zone in Euler coordinates. The boundary of the ellipse Γ_t has the equation $\tau^{-2}x^2 + \tau^2 y^2 = 1$. The point $(x = \tau,\ y = 0)$ in the xy plane corresponds to the point $(\xi = 1,\ \eta = 0)$.

Let us estimate the distance Δx along the axis from the end of the semimajor axis of the ellipse $x = \tau$, $y = 0$, to the points on Γ_t, where $\theta \sim t^{-1}$. Taking into account that

$$\theta = \operatorname{arc\,tg}(\eta / \xi) = \operatorname{arc\,tg}(\tau^2 y / x)$$

we find

$$\Delta x = O(t^{-1}) \quad \text{for} \quad t \to \infty \tag{3.12}$$

Thus, in the case of the c-solution the free-boundary instability is localized near the ends of the major axis. The size of the instability zone is estimated by the relationship (3.12). In particular, we apply this deduction to the problem of stability of the motion (1.1) with an impermeable wall $\eta = 0$. As $t \to \infty$ the fluid is squeezed to the wall, and this stabilizes the free boundary outside the mentioned instability zones. A similar stabilizing effect has been detected in [1, 3] in investigations of the simpler problem of stability of a liquid bar under a stamp.

Examining the s-solution, we conclude that the quantity increases according to the law $R_s \sim t^{\delta}$ at a distance $\Delta x = O(t^{1-2\delta})$, $0 \le \delta \le 2$ from the ends of the major axis of the ellipse. In this case the free-boundary instability domain increases without limit in the xy plane as time elapses. (However, let us note that the maximum instability domain, where $R_s \sim t^2$, diminishes in proportion to t^{-3}.) This result is even more interesting since their tendency to zero in a potential metric was proved above for the s-solution as $t \to \infty$.

4. PROOF OF PROPOSITION 2

The validity of (3.4), (3.5) for any fixed n results substantially from (3.1). In order to prove the dimensionality of these estimates relative to n, the behavior of solutions of the Cauchy problem (2.13), (2.14) should be studied as $n \to \infty$.

Let us examine the first of (2.14). The coefficient of a_n in this equation is the sum of two components. The first component will be the principal one for $n \to \infty$ and $\tau \ll \sqrt{n}$, and the second for $\tau \gg \sqrt{n}$.

Hence, the solution of the Cauchy problem is considered first for a_n in the interval $[1, \alpha\sqrt{n}]$; then the solution is continued into the interval $[\alpha\sqrt{n}, \infty)$. We select the constant $\alpha > 0$ below.

Reasoning used to study the asymptotic of eigenfunctions of the Sturm-Liouville problem (see [5], say) is used in analyzing the solution for large n and $1 \le \tau \le \alpha\sqrt{n}$. Let

$$Q = 2(\tau^4 + 1)^{-1} \operatorname{th} ns(\tau) \tag{4.1}$$

and let us introduce new variables by using the substitution

$$\sigma = \int_1^{\tau} [Q(\zeta)]^{1/2} d\zeta, \qquad \omega = [Q(\tau)]^{1/4} a_n \tag{4.2}$$

This substitution converts the interval $1 \le \tau \le \alpha\sqrt{n}$ into the interval $0 \le \sigma \le l_n$, and (2.14) for a_n into

$$d^2\omega / d\sigma^2 + n\omega = \rho(\sigma)\,\omega \tag{4.3}$$

$$\rho(\sigma) = \frac{Q''}{4Q^2} - \frac{5Q^2}{16Q^3} - \frac{r}{Q} \tag{4.4}$$

The right side of (4.4) is considered as a function of σ. The function $\rho(\sigma)$ is continuous in the interval $[0, l_n]$ and because of (4.1), (4.2) admits of the estimate $|\rho| \leq C_4\tau^2$ for any natural n. Here $\tau(\sigma)$ is a function defined first from (4.1). On the basis of (4.1), (4.2) we have

$$\tau(l_n) = \alpha\sqrt{n}, \quad \tau^{-2} d\sigma / d\tau \geqslant 1 \quad \text{where} \quad 0 \leqslant \sigma \leqslant l_n$$

Hence

$$\tau \leqslant \alpha\sqrt{n}\,[1 + \alpha\sqrt{n}\,(l_n - \sigma)]^{-1}$$

This results in an explicit estimate of $|\rho|$ as a function of σ

$$|\rho| \leqslant C_4\alpha^2 n\,[1 + \alpha\sqrt{n}\,(l_n - \sigma)]^{-2} \tag{4.5}$$

Let us turn to (4.3). Conditions (2.14) generate initial conditions for this equation

$$\omega(0) = 1, \quad \omega'(0) = 0$$

The solution of (4.3) with these initial conditions satisfies the Volterra integral equation [5]

$$\omega(\sigma) = \cos n^{1/2}\sigma + n^{-1/2}\int_0^\sigma \sin n^{1/2}(\sigma - \zeta)\,\rho(\zeta)\,\omega(\zeta)\,d\zeta \tag{4.6}$$

Applying successive approximations, we obtain the solution of (4.6) in the form $\omega = \lim \omega_k$, $k \to \infty$, where $\omega_0 = \cos n^{1/2}\sigma$, and the function ω_k is defined by (4.6) for $k \geq 1$, in whose right side ω is replaced by ω_{k-1}. Using (4.5), we find the following estimate for the kernel of (4.6):

$$\left| n^{-1/2}\int_0^\sigma \rho(\zeta)\sin n^{1/2}(\sigma - \zeta)\,d\zeta \right| < 2C_4\alpha$$

Let us select α less than $1/2C_4$, and let us fix it. Then the successive approximations will converge uniformly to the solution $\omega(\sigma)$ of (4.6) for all $\sigma \in [0, l_n]$ and sufficiently large n, where the function ω is uniformly bounded for $0 \leq \sigma \leq l_n$, $n \to \infty$.

In conjunction with (4.2) this results in the estimate $|a_n| \leq c_5\tau$ for $1 \leq \tau \leq \alpha\sqrt{n}$. The estimate for a_n' is obtained by differentiating (4.2) and (4.6). It is

$$|a_n'| \leqslant C_5 n^{1/2}\tau^{-1}$$

Now, let us examine the first equation in (2.14) in the interval $[\alpha, \sqrt{n}, \infty)$. Let us make the change in variable $a_n = \sqrt{n}A_n$, $\tau = \sqrt{n}\zeta$ in this equation, and let us extract the main term in the coefficient of the equation obtained for A_n. We will have

$$\frac{d^2A_n}{d\zeta^2} + \left[-\frac{2}{\zeta^2} + \frac{2}{\zeta^4}\,\mathrm{th}\,\frac{1}{\zeta^2} + O(n^{-2}\zeta^{-})\right]A_n = 0 \quad \text{for} \quad \zeta \geqslant \alpha, \quad n \to \infty \tag{4.7}$$

Let us pose the Cauchy problem for this equation

$$A_n = n^{-1/2}a_n(\alpha n^{1/2}), \quad dA_n / d\zeta = a_n'(\alpha n^{1/2}) \quad \text{for} \quad \zeta = \alpha$$

From the estimates obtained earlier for a_n, a_n' for $1 \leq \tau \leq \alpha\sqrt{n}$, we conclude that $A_n(\alpha)$, $dA_n(\alpha)/d\zeta$ are uniformly bounded in n as $n \to \infty$. There hence results that the solution of the Cauchy problem for (4.7) is estimated thus:

$$|A_n| \leqslant C_6\zeta^2, \quad |dA_n / d\zeta| \leqslant C_6\zeta \quad \text{for} \quad \zeta \geqslant \alpha$$

Now going over to the variables a_n, τ, we obtain

$$|a_n| \leqslant C_6\tau^2 / \sqrt{n}, \quad |a_n'| \leqslant C_6\tau / \sqrt{n} \quad \text{for} \quad \tau \geqslant \alpha\sqrt{n}$$

The inequalities (3.4), (3.5) for a_n are proved.

The behavior of the solutions of the second of Eqs. (2.14) with the initial conditions (2.13) is investigated analogously for large n. The difference is only that in the last stage the equation

$$\frac{d^2B_n}{d\zeta^2} + \left[-\frac{2}{\zeta^2} + \frac{2}{\zeta^4}\,\mathrm{cth}\,\frac{1}{\zeta^2} + O(n^{-2}\zeta^{-6})\right]B_n = 0$$

is obtained in place of (4.7) for $B_n = n^{-1/2} b_n$, and for its Cauchy problem with the values $B_n(\alpha)$, $dB_n(\alpha)/d\zeta$ bounded as $n \to \infty$. The inequalities

$$|B_n| \leqslant C_7 \zeta, \quad |dB_n/d\zeta| \leqslant C_7 \quad \text{for} \quad \zeta \geqslant \alpha$$

are valid for B_n.

Consequently, we obtain the estimates

$$|b_n| \leqslant C_7 \tau, \quad |b_n'| \leqslant C_7 \quad \text{for} \quad \tau \geqslant \alpha \sqrt{n}$$

This completes the proof of Proposition 2.

The author is grateful to L. V.Ovsyannikov and R. M. Garipov for discussing the research.

LITERATURE CITED

1. L. V. Ovsyannikov, "General equations and examples." in: Problems of Unsteady Fluid Motion with a Free Boundary [in Russian]. Nauka, Novosibirsk (1967).
2. V. M. Kuznetsov and E. N. Sher, "On the stability of ideal incompressible fluid flow in a strip and a ring," Zh. Prikl. Mekh. i Tekh. Fiz., No. 2 (1964).
3. L. V. Ovsyannikov, "On the disturbances of an unsteady motion of liquid with free boundary," Fluid Dynamics Trans., Vol. 4, Warszawa, Panstwowe Wydawn. Naukowe (1969).
4. V. I. Smirnov, Course in Higher Mathematics [in Russian], Vol. 5, Fizmatgiz, Moscow (1959).
5. I. G. Petrovskii, Lectures on Partial Differential Equations [in Russian], Fizmatgiz, Moscow (1961).